Symmetry properties in surface growth models

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An analysis of the consequences of the symmetry properties for the Kardar-Parisi-Zhang equation is presented. The formalism of the effective dynamical action for the differential equation in the presence of white noise is reviewed and used to derive a set of Ward identities. The analysis is first applied to the problem of the renormalizability of the theory in $d = 2 + 1$. A set of Ward identities for the classical Galileian invariance, taking into account the boundary conditions of the problem, is derived and applied to obtain exact relations for the renormalized physical quantities.

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I. INTRODUCTION

In the last few years there has been considerable effort in understanding the behavior of growing interfaces in the presence of disorder [1]. The subject has attracted a lot of attention for the experimental relevance of the problem and also, from the theoretical point of view, because of the wide range of different physical phenomena described by the corresponding nonlinear dynamics.

When overhanging can be neglected and when the growth process is governed by local conditions, dynamics of an isotropically growing interface in the presence of disorder can be described by a stochastic nonlinear differential equation, the so-called Kardar-Parisi-Zhang (KPZ) equation [2]. From a phenomenological point of view the equation follows directly from the symmetry properties of the physical problem: invariance under translations of the height of the surface, isotropy in space, lack of parity under spatial inversion, growth in a direction perpendicular to the surface. Choosing a d dimensional reference plane, we can measure the height h at the point $\mathbf{x} \in \mathbb{R}^d$ at time t. The evolution of $h(\mathbf{x}, t)$ is then described by the stochastic differential equation

$$\dot{h}(x,t) = \nu \Delta h(x,t) + \frac{\lambda}{2}[\nabla h(x,t)]^2 + \eta(x,t), \tag{1}$$

where $\eta(x,t)$ is a Gaussian noise, with zero mean after the subtraction of the average growth velocity of the surface:

$$\begin{aligned} \langle \eta(x,t) \rangle &= 0, \\ \langle \eta(x,t)\eta(y,\tau) \rangle &= D\delta^d(x-y)\delta(t-\tau). \end{aligned} \tag{2}$$

Equation (1) is the simplest nonlinear generalization of the diffusion equation. It is closely related to many different physical situations, such as flame-front propagation [3], driven diffusion [4], and the behavior of magnetic flux lines in superconductors [5]. Moreover, Eq. (1) can be mapped, with very simple transformations, into the Burgers equation, describing an infinitely compressible irrotational fluid [6], or into a linear equation describing a static problem for the configurational energy of a polymer in quenched disorder [7, 8]. The simple nonlinear force acting on the system is the origin of the wide applicability. Nevertheless, the effects of the nonlinear term are difficult to treat: as an example, this term is not of the potential type, i.e., it does not correspond to a functional derivative with respect to $h(\mathbf{x}, t)$ of a Hamiltonian:

$$(\boldsymbol{\nabla} h)^2 \neq \frac{\delta \mathcal{H}(h)}{\delta h}. \tag{3}$$

The explicit solution of (1) is known in the ordered case [6]. Choosing a constant initial condition, the solution gives rise to the well known parabola patterns for the surface, common to many physical situations.

The solution is, in general, not known in the presence of noise. Nevertheless, the system shows, in the ordered and in the disordered case, an interesting scaling behavior for large space and time scales [2]. The scaling properties are described by power laws depending on two critical exponents, namely, the scaling of relaxation times with length,

$$\delta x \sim t^{\frac{1}{z}}, \tag{4}$$

and the mean square fluctuation w,

$$w = \langle |h(\mathbf{x},t) - h(\mathbf{x}',t')|^2 \rangle^{\frac{1}{2}}, \tag{5}$$

of the height of the surface at two points at distance L in the large time regime:

$$w \propto L^{\chi} f\left[\frac{t}{L^z}\right], \tag{6}$$

where the function $f(x)$ is constant at large values of x and for small x behaves as $x^{\chi/z}$. In the ordered case it is simple to show that the two exponents are not independent. Indeed we have

$$\chi + z = 2. \tag{7}$$

The effects of disorder on the roughening of the surface are investigated here by studying the modifications of the critical exponents with respect to the deterministic case. There exist a few exact results about the critical exponents in the presence of disorder.

They are exactly known in $d = 1$ because of relation

(7), still holding, and because of a fluctuation-dissipation theorem holding in $d = 1$ [9]. Analytic calculations on a Cayley tree give a simple diffusion behavior in $d = \infty$ [10]. For the intermediate region, the upper critical dimension of the model is still not known and different conjectures describing the dependence of the critical exponents on d have been proposed [11]. Another approach is based on dynamical renormalization group calculations, using an expansion in the nonlinear coupling constant and in the noise strength [12]. These calculations lead to the conclusion that, in spite of the scaling properties in the long-wavelength limit, the system has a nonperturbative behavior in the physical case $d = 2$ and a regime of "asymptotic freedom" is present. The cancellations of divergence in the perturbative expansion are probably due to symmetry relations holding between correlation functions.

In this scenario exact results concerning the physical quantities of the system, such as correlation and response functions, would be of great interest and it is believed that symmetries play a central role in the investigation of the scaling properties of the model. In this paper we will therefore analyze in detail the consequences of the symmetry properties of the KPZ equation.

The paper is organized as follows. In Sec. II we review in detail the path integral formulation for the KPZ equation. Using this formalism, in Sec. III we discuss the problem of the renormalizability for the dynamical action. In Sec. IV we analyze the whole symmetry group of the KPZ equation, pointing out the existence of a kind of *conformal symmetry* in the case of a Gaussian noise and deriving the effect of symmetries on the renormalizability of the theory in $d = 2$. The Ward identities for the Galileian invariance of the system will be used to derive directly the invariance under the renormalization procedure of the coupling constant for the nonlinear term and the fundamental relation between the two critical exponents z and χ in Sec. V. In Sec. VI the effects of boundary conditions will be taken into account to derive some preliminary consequences of the Galileian invariance on the renormalized physical quantities.

II. THE EFFECTIVE ACTION FOR THE KPZ EQUATION

Let us consider a stochastic differential equation for a scalar field $h(x,t)$, as a function of a spatial variable $x \in \mathbb{R}^d$ and of the time t:

$$F[h(x,t),\eta(x,t)] = 0, \tag{8}$$

where $\eta(x,t)$ is a noise with given probability distribution

$$[d\sigma(\eta)] = [d\eta]\exp[-\rho(\eta)]. \tag{9}$$

For Langevin type equations, if $F[h,\eta]$ has no singularities, the stochastic field $h_\eta(x,t)$ represents the unique solution of Eq. (8). A generic functional depending on $h_\eta(x,t)$ can be formally written in terms of a path integral:

$$A(h_\eta) = \int [dh]\delta(h - h_\eta)A(h), \tag{10}$$

where $[dh]$ is the usual definition for the measure in the functional integral and $\delta(h - h_\eta)$ is a formal expression for the Dirac functional. With a change of variable (10) can be written as

$$A(h_\eta) = \int [dh]\delta(F[h,\eta])A(h)\det M, \tag{11}$$

where

$$M(x,t;y,\tau) = \frac{\delta F[h(x,t),\eta(x,t)]}{\delta h(y,\tau)} \tag{12}$$

is the Jacobian of the transformation. Expression (11) can be restated by introducing an effective action for the differential equation (8), the so-called *dynamical action*, whose variation gives an equation of motion for h corresponding to (8). Let us write the determinant of the transformation in (12) as an integral over two Grassmann fields C and $\bar{C}$:

$$\det M = \int [d\bar{C}][dC]\exp\left(\int d^dxdtd^dyd\tau\,\bar{C}(x,t)\times M(x,t;y,\tau)C(y,\tau)\right) \tag{13}$$

and let us choose a formal expression for the functional δ by introducing an auxiliary field $\hat{h}$:

$$\delta(F[h,\eta]) = \int [d\hat{h}]\exp\left(-\int d^dxdt\quad \hat{h}(x,t)\times F[h(x,t),\eta(x,t)]\right). \tag{14}$$

With these positions expression (11) can be written as

$$A(h_\eta) = \int [dh][d\hat{h}][d\bar{C}][dC]A(h)\times\exp\{-[S(h,\hat{h},C,\bar{C},\eta)]\}, \tag{15}$$

where now S is the effective dynamical action [13] for the differential equation (8) (the notation for functional relationship is standard):

$$S(h,\hat{h},C,\bar{C},\eta) = \int d^dxdt\hat{h}(x,t)F[h(x,t),\eta(x,t)] - \int d^dxdtd^dyd\tau\bar{C}(x,t)\times M(x,t;y,\tau)C(y,\tau). \tag{16}$$

We are usually interested in averaging physical quantities such as A over the distribution of the noise (9). Using the formalism of the effective action the average can be explicitly taken. As an example let us consider the generating functional $Z(J)_\eta$ for the correlation functions of the field $h(x,t)$:

$$Z(J)_\eta = \exp\left(\int \{d^dxdt\quad J(x,t)h_\eta(x,t)\}\right) = \int [dh][d\hat{h}][d\bar{C}][dC]\exp\{-[S(h,\hat{h},C,\bar{C},\eta)] + \int d^dxdt\quad J(x,t)h(x,t)\}. \tag{17}$$

If the differential equation is linear in the noise:

$$F[h(x,t)] = \eta(x,t), \tag{18}$$

$Z(J)_\eta$ can be explicitly averaged over the noise distribution:

$$\langle Z(J)\rangle = \int [d\sigma(\eta)] \frac{Z(J)_\eta}{Z(0)_\eta}. \tag{19}$$

Using (18) we obtain

$$\langle Z(J)\rangle = \int [dh][d\hat{h}][d\bar{C}][dC] \exp\{-[S(h,\hat{h},C,\bar{C})] + \int d^dx dt\ J(x,t)h(x,t)\}, \tag{20}$$

where now S is given by

$$S(h,\hat{h},C,\bar{C}) = -w(\hat{h}) + \int d^dxdt\, \hat{h}(x,t)F[h(x,t)] - \int d^dxdt\ d^dyd\tau\ \bar{C}(x,t) \times M(x,t;y,\tau)C(y,\tau), \tag{21}$$

and $w(\hat{h})$ is the Laplace transform of the noise distribution (9):

$$w(\hat{h}) = \int [d\sigma(\eta)] \exp\left(\int d^dxdt\ \ \hat{h}(x,t)\eta(x,t)\right). \tag{22}$$

Notice that the equivalence between the fermionic integral and the determinant depends on the boundary condition chosen for the fermionic fields [14].

The formalism of the dynamical action is heavy when compared to the usual approach based on the direct analysis of the stochastic differential equation. Nevertheless, it has very interesting consequences. First, it allows an explicit integration over the noise directly on the partition function. Moreover, the dynamical action allows one to restate in the language of quantum field theory a dynamical model described by the differential equation (8). This equivalence is of particular interest because it entails the extension of all perturbative and exact techniques developed in path integral formalism to the case of a stochastic dynamical problem. As will be clear in the following analysis, the dynamical action offers a natural background for the study of the symmetry properties of a stochastic differential equation.

The relevant physical quantities can be rewritten in the language of the dynamical action. In particular, the auxiliary field $\hat{h}$ has a physical meaning: the correlation and response functions can be generated by taking derivatives of the free energy with respect to the fields h and $\hat{h}$. It is easy to see that the two point correlation function $\mathcal{C}$ and the response function $\mathcal{G}$ of the system are given by

$$\mathcal{C}(x,t;y,\tau) = \langle h(x,t)h(y,\tau)\rangle = \left\langle \frac{\delta^2 W(J)}{\delta J_h(x,t)\delta J_h(y,\tau)} \right\rangle \tag{23}$$

and

$$\mathcal{G}(x,t;y,\tau) = \langle h(x,t)\hat{h}(y,\tau)\rangle = \left\langle \frac{\delta^2 W(J)}{\delta J_h(x,t)\delta J_{\hat{h}}(y,\tau)} \right\rangle, \tag{24}$$

where W is the free energy and J_h and $J_{\hat{h}}$ are the conjugated fields for h and $\hat{h}$, as will be explained in (27) and (25) below.

The dynamical action for the KPZ equation (1) with Gaussian noise (2) can be easily obtained. By integrating over the stochastic field η the partition function and the free energy of the model can be expressed as in (20):

$$Z(J_h,J_{\hat{h}},J_C,J_{\bar{C}}) = \int [dh][d\hat{h}][d\bar{C}][dC] \exp\Big\{-[S(h,\hat{h},C,\bar{C})] + \int d^dxdt\ \ J_h(x,t)h(x,t) + J_{\hat{h}}(x,t)\hat{h}(x,t) + J_C(x,t)C(x,t) + J_{\bar{C}}(x,t)\bar{C}(x,t)\Big\}, \tag{25}$$

where now S is the dynamical action for the KPZ equation:

$$S(h,\hat{h},C,\bar{C}) = \int d^dxdt\Big\{-\frac{D}{2}\hat{h}(x,t)^2 + \hat{h}(x,t)\left[\frac{\partial}{\partial t}h(x,t) - \nu\Delta h(x,t) - \frac{\lambda}{2}[\nabla h(x,t)]^2\right] - \bar{C}(x,t)\left[\frac{\partial}{\partial t}C(x,t) - \nu\Delta C(x,t) - \lambda\nabla h(x,t)\nabla C(x,t)\right]\Big\}. \tag{26}$$

A corresponding relation holds for the generating functional of connected correlation functions:

$$W(J_h,J_{\hat{h}},J_C,J_{\bar{C}}) = \langle \ln Z(J_h,J_{\hat{h}},J_C,J_{\bar{C}})_\eta\rangle_\eta, \tag{27}$$

and for the thermodynamic potential Γ expressed as a function of the mean values of the fields; setting

$$h_M(x,t) = \left\langle \frac{\delta W(J)}{\delta J_h(x,t)} \right\rangle \tag{28}$$

and the corresponding relations for the fields $\hat{h}$, C, $\bar{C}$, the Legendre transformation on W gives

$$\Gamma(h_M,\hat{h}_M,C_M,\bar{C}_M) = W(J_h,J_{\hat{h}},J_C,J_{\bar{C}}) - J_h h_M - J_{\hat{h}}\hat{h}_M - J_C C_M - J_{\bar{C}}\bar{C}_M. \tag{29}$$

Let us recall the symmetry properties of the dynamical action S, without looking at the physical problem described by the stochastic differential equation. The dynamical action is invariant under the supersymmetry transformation [15]

$$\begin{aligned} \delta h &= \bar{\epsilon} C, \quad \delta \hat{h} = 0, \\ \delta C &= 0, \quad \delta \bar{C} = \bar{\epsilon} \hat{h}, \end{aligned} \tag{30}$$

where $\bar{\epsilon}$ is a Grassmann variable. The Becchi-Rouet-Stora (BRS) transformation (30), first introduced in quantized field theories [15], corresponds here to the linearized version of the invariance of the functional integral with respect to a translation of the stochastic field η. The noise is a variable of integration and can therefore be shifted without affecting the value of the integral.

III. POWER COUNTING AND RENORMALIZATION FOR THE KPZ EQUATION

The dynamical problem described by the KPZ equation can be restated in terms of a quantum field theory by introducing the formalism of the dynamical action. In this framework, the generating functionals for the correlation and response functions can be computed in a standard way using a perturbative approach in some small parameter, at least in principle. We will first consider a formal approach, without taking into account the problem of asymptotic freedom. The perturbative approach requires in any case the study of the renormalizability of the theory.

It is useful to consider a more general noise depending on the field h:

$$\begin{aligned} \dot{h}(x,t) = \nu \Delta h(x,t) + \frac{\lambda}{2}[\Delta h(x,t)]^2 \\ + a[h(x,t)]\eta(x,t), \end{aligned} \tag{31}$$

where $a[h(x,t)]$ is a generic function depending on the field h and on its derivatives and η is the Gaussian noise (2). We are always forced to consider a Gaussian noise: otherwise we would introduce nonlocal terms in the dynamical action. In this case the explicit form of S is

$$\begin{aligned} S(h,\hat{h},C,\bar{C}) = \int d^dx dt - \frac{D}{2}\left[\hat{h}(x,t)a[h(x,t)] - \int d^dy d\tau \bar{C}(x,t)\frac{\delta a[h(x,t)]}{\delta h(y,\tau)}C(y,\tau)\right]^2 \\ + \int d^dx dt \hat{h}(x,t)\left[\frac{\partial}{\partial t}h(x,t) - \nu\Delta h(x,t) - \frac{\lambda}{2}[\nabla h(x,t)]^2\right] \\ -\bar{C}(x,t)\left[\frac{\partial}{\partial t}C(x,t) - \nu\Delta C(x,t) - \lambda \nabla h(x,t)\nabla C(x,t)\right]. \end{aligned} \tag{32}$$

Let us now compute the renormalizability dimension of the theory by power counting [16]. Frequency has dimension 2 in order to obtain a homogeneous propagator at large impulses. From the dimension of the integration measure in the action (26) we compute the canonical dimension of the fields:

$$\begin{aligned} [h] &= \frac{1}{2}(d-2) + [a], \\ \left[\hat{h}\right] &= \frac{1}{2}(d+2) - [a], \\ \left[C + \bar{C}\right] &= d. \end{aligned} \tag{33}$$

The canonical dimension of the operator corresponding to the nonlinear term in the differential equation implies that the theory is renormalizable by power counting if the field h is dimensionless:

$$[F_{int}] = \frac{1}{2}(d+2) + [a] = 0. \tag{34}$$

When a is a constant as in (1), condition (34) implies that the theory is renormalizable by power counting if

$$d = 2. \tag{35}$$

Condition (35) does not imply the renormalizability of the theory. The procedure of the renormalization group could introduce new operators in the dynamical action even in $d = 2$. In the case of interaction terms coming from a functional derivative of a static action, the renormalizability of the theory can be deduced from the supersymmetry transformations (30) and from a new independent supersymmetry [16]. These also imply that the dynamic correlation functions at large time scales converge to the corresponding static ones, computed with the static action.

Here the dynamical action has the simple supersymmetry (30) and this is not enough to ensure the renormalizability of the model in $d = 2$. In this case the symmetry of the physical problem plays a fundamental role to provide new constraints for the operators in the renormalized action.

Let us now study the case a=const, and therefore $[a] = 0$. A generic operator O in the renormalized action contains n spatial derivatives, n' temporal derivatives, k fields $\hat{h}$, k' fields $\bar{C}C$, and m fields h. From the dimension of the integration measure we can deduce that the theory is renormalizable by power counting if

$$\left(\int d^2x dt O\right) \geq 0, \tag{36}$$

implying

$$4 - n - 2n' - 2k - 2k' \geq 0. \tag{37}$$

The number m of fields h is not fixed since they are adimensional in $d = 2$. It is easy to construct the set of operators whose dimension is compatible with (37). Apart from the ones corresponding to the starting dynamical action:

$$\hat{h}\hat{h},\quad \hat{h}\frac{\partial h}{\partial t},\quad \hat{h}\Delta h,\quad \hat{h}(\nabla h)^2,$$
$$\frac{\partial(\bar{C}C)}{\partial t},\quad \Delta(\bar{C}C),\quad \nabla h\nabla(\bar{C}C), \tag{38}$$

we can have additional terms with derivatives in h,

$$\frac{\partial h}{\partial t},\quad \left(\frac{\partial h}{\partial t}\right)^2,\quad \frac{\partial^2 h}{\partial t^2},\quad \Delta h\frac{\partial h}{\partial t},\quad (\nabla h)^2\frac{\partial h}{\partial t},$$
$$(\nabla h)^2,\quad \Delta h,\quad (\nabla h)^4,\quad (\Delta h)^2,\quad \Delta h(\nabla h)^2, \tag{39}$$

terms in $\bar{C}C\ h$:

$$\frac{\partial h}{\partial t}\bar{C}C,\quad \Delta h\bar{C}C,\quad (\nabla h)^2\bar{C}C, \tag{40}$$

and terms in $\hat{h}$ and $\bar{C}C$ and h:

$$\frac{\partial \hat{h}}{\partial t},\quad \hat{h},\quad \Delta\hat{h},\quad \nabla h\nabla\hat{h},$$
$$\bar{C}C,\quad \bar{C}C\bar{C}C,\quad \hat{h}\bar{C}C. \tag{41}$$

In every term an arbitrary power of the adimensional field h can be present. The form of the starting dynamical action is not conserved if we do not introduce some constraints on the renormalized operators. We will therefore consider the symmetry properties of the dynamical problem.

IV. SYMMETRY PROPERTIES OF THE KPZ EQUATION

Let us now consider the physical problem described by the KPZ equation. From the derivation of Sec. I we can easily obtain the complete symmetry group of the differential equation, i.e., the transformation of spatial and time variables, of the field h and of its derivatives that maps solutions of Eq. (1) into solutions, leaving the correlations of the stochastic field η unchanged [18]. The KPZ equation with noise (2) is invariant under translations in space:

$$T(\mathbf{x},t,h,\eta) = (\mathbf{x}+\epsilon,t,h,\eta),, \tag{42}$$

translations in time:

$$T(\mathbf{x},t,h,\eta) = (\mathbf{x},t+\epsilon,h,\eta), \tag{43}$$

rotations in space:

$$T(\mathbf{x},t,h,\eta) = (\mathcal{R}\mathbf{x},t,h,\eta), \tag{44}$$

translations of the field:

$$T(\mathbf{x},t,h,\eta) = (\mathbf{x},t,h+\epsilon,\eta), \tag{45}$$

scale transformations:

$$T(\mathbf{x},t,h,\eta) = (e^{\epsilon}\mathbf{x},e^{2\epsilon}t,h,e^{2\epsilon}\eta) \tag{46}$$

[the KPZ equation with white noise is invariant under transformation (46) in $d=2$], Galileian transformations:

$$T(\mathbf{x},t,h,\eta) = \left(\mathbf{x}+\lambda\epsilon t,t,h+\mathbf{x}\cdot\epsilon+\frac{\lambda}{2}\epsilon^2 t,\eta\right) \tag{47}$$

and "local" transformations:

$$T(\mathbf{x},t,h,\eta) = \left(\frac{\mathbf{x}}{1+2\lambda\epsilon t},\ \frac{t}{1+2\lambda\epsilon t},\ h-\frac{d\nu}{\lambda}\ln(1+2\lambda\epsilon t)-\frac{\epsilon x^2}{1+2\lambda\epsilon t},\ \frac{\eta}{(1+2\lambda\epsilon t)^2}\right). \tag{48}$$

The KPZ equation with white noise is invariant under this interesting transformation in $d=2$.

Let us now consider the consequences of the symmetry group of Eq. (1) on the renormalized dynamical action S_R. The scaling symmetry (46) and the "local" symmetry (48) are not conserved by the introduction of an ultraviolet cutoff and they generate anomalies in the theory. The remaining symmetries are conserved by the renormalization procedure and they can be used to obtain constraints on the renormalized operators. Taking into account the operators compatible with power counting listed in Sec. III, the most general renormalized action can be written as

$$S_R(h,\hat{h},C,\bar{C}) = \Sigma(h,\hat{h}) - \int d^dx dt d^dy d\tau\ \ C(\bar{x},t) \times M_{h,\hat{h},C,\bar{C}}(x,t;y,\tau)C(y,\tau). \tag{49}$$

Arbitrary powers of the dimensionless field, such as h^n, are not allowed by the invariance of the equation under *translations of the field.* Moreover, the *rotational invariance* rules out single gradient terms.

From power counting arguments, the only operators in the fields $\hat{h}$ and $\bar{C}C$ allowed in the matrix M are

$$a\hat{h}\bar{C}C \tag{50}$$

and

$$b\bar{C}C\bar{C}C. \tag{51}$$

Both terms can be ruled out in $d>2$ [16]. Here we have to take them into account explicitly.

Let us now consider the consequences of the BRS transformation (30). The symmetry is conserved by the renormalization procedure. It is possible to derive the corresponding Ward identities for the renormalized dynamical action:

$$\int d^dx dt \left[C_M(x,t)\frac{\delta}{\delta\ h_M(x,t)} + \hat{h}_M(x,t)\frac{\delta}{\delta\bar{C}_M(x,t)}\right] \times S_R(h_M,\hat{h}_M,C_M,\bar{C}_M). \tag{52}$$

Derivatives with respect to $\hat{h}_M$, C_M and then h_M, $\bar{C}_M$ give a relation between the coefficients of the operators in (50) and (51):

$$\frac{\delta a}{\delta h_M} = -2b. \tag{53}$$

From translation invariance in h, a depends on h only through its derivatives and this rules out (50) and (51) because of power counting. We therefore obtain

$$M_{h_M,\hat{h}_M,C,\bar{C}_M}(x,t;y,\tau) = M_{h_M}(x,t;y,\tau). \tag{54}$$

Substituting in (52) the explicit expression for the renormalized dynamical action (49) with the condition (54), we obtain two relations for the functionals $\Sigma(h_M,\hat{h}_M)$ and $M(h_M)$:

$$\frac{\delta}{\delta h_M(y,\tau)}M_{h_M}(x,t;z,\sigma)+\frac{\delta}{\delta h_M(z,\sigma)}M_{h_M}(x,t;y,\tau)=0 \tag{55}$$

and

$$\frac{\delta}{\delta h_M(x,t)}\Sigma(h_M,\hat{h}_M)=\int d^dyd\tau\hat{h}_M(y,\tau)M_{h_M}(y,\tau;x,t). \tag{56}$$

Condition (55) implies

$$M_{h_M}(x,t;y,\tau)=\frac{\delta F[h(x,t)]}{\delta h_M(y,\tau)}. \tag{57}$$

Relation (55) represents the integration condition for Eq. (56). This can be solved and gives [16]

$$\Sigma(h_M,\hat{h}_M)=\int d^dyd\tau \quad \hat{h}_M F[h_M]+w(\hat{h}_M). \tag{58}$$

It is now easy to see that the only terms compatible with (58) are the ones already present in the starting Hamiltonian. Derivatives of the BRS Ward identities (52) give the standard relations between the coefficients of the operators in the fermionic and bosonic parts. The original form of the Hamiltonian is thus conserved:

$$S_R(h_M,\hat{h}_M,C_M,\bar{C}_M)=\int d^dxdt-\frac{a}{2}[\hat{h}_M(x,t)]^2+\hat{h}_M(x,t)\left(b\frac{\partial}{\partial t}h_M(x,t)-c\Delta h_M(x,t)-\frac{d}{2}[\nabla h_M(x,t)]^2\right)$$
$$-\bar{C}_M(x,t)\left(b\frac{\partial}{\partial t}C_M(x,t)-c\Delta C_M(x,t)-d\nabla h_M(x,t)\nabla C_M(x,t)\right), \tag{59}$$

where a, b, c and d are arbitrary constants.

The heuristic demonstration is the formal starting point to derive the effects of the whole symmetry group on the renormalized Hamiltonian. As an example we will analyze explicitly the consequences of the Galileian invariance in the KPZ equation.

V. WARD IDENTITIES FROM GALILEIAN INVARIANCE

The symmetry properties of a generic differential equation

$$F[x,t,\eta,h]=0 \tag{60}$$

under a transformation T of the fields and of the space-time variables imply that

$$F[x_T,t_T,\eta_T,h_T]=F[x,t,\eta,h]. \tag{61}$$

The effective dynamical action contains the new fields $\hat{h}$, $\bar{C}$, and C. The symmetry properties of Eq. (1) imply the corresponding transformation laws for the additional fields. These are obtained by setting

$$S(h_T,\hat{h}_T,C_T,\bar{C}_T,\eta_T)=S(h,\hat{h},C,\bar{C},\eta) \tag{62}$$

and deriving the corresponding transformation for the fields from the invariance of the action. Let us now take into account the invariance of the KPZ equation under Galileian transformations. It is well known that Eq. (1) can be mapped into the hydrodynamic Burgers equation [8], by setting

$$\mathbf{v}=-\nabla h \tag{63}$$

and in this framework Galileian invariance corresponds to a translation in the velocity field $\mathbf{v}$. For the KPZ equation, the transformation corresponds to a tilting of the surface of a small angle ϵ. By using (62) and the invariance of the measure, so that $d^dx_Tdt_T=d^dxdt$, one can easily obtain the transformations for the auxiliary fermionic and bosonic fields. The Ward identities for this transformation simply come from the fact that the fields are variables of integration in the functional integral and they can be changed without affecting the value of the integral. Writing an explicit expansion in ϵ for the transformations of the fields we easily get the Ward identities for the free energy W:

$$\int d^dxdt\Bigg\{J_h(x,t)\frac{\partial}{\partial x_\mu}\epsilon_\mu\frac{\delta W}{\delta J_h(x,t)}\lambda t+J_C(x,t)\frac{\partial}{\partial x_\mu}\epsilon_\mu\frac{\delta W}{\delta J_C(x,t)}\lambda t$$
$$+J_{\hat{h}}(x,t)\frac{\partial}{\partial x_\mu}\epsilon_\mu\frac{\delta W}{\delta J_{\hat{h}}(x,t)}\lambda t+J_{\bar{C}}(x,t)\frac{\partial}{\partial x_\mu}\epsilon_\mu\frac{\delta W}{\delta J_{\bar{C}}(x,t)}\lambda t+J_h(x,t)x_\mu\epsilon_\mu\Bigg\}=0, \tag{64}$$

and for the thermodynamic potential Γ:

$$\int d^dxdt\left\{h_M(x,t)\frac{\partial}{\partial x_\mu}\epsilon_\mu\frac{\delta\Gamma}{\delta h_M(x,t)}\lambda t+C_M(x,t)\frac{\partial}{\partial x_\mu}\epsilon_\mu\frac{\delta\Gamma}{\delta C_M(x,t)}\lambda t+\hat{h}_M(x,t)\frac{\partial}{\partial x_\mu}\epsilon_\mu\frac{\delta\Gamma}{\delta\hat{h}_M(x,t)}\lambda t\right.$$
$$\left.+\bar{C}_M(x,t)\frac{\partial}{\partial x_\mu}\epsilon_\mu\frac{\delta\Gamma}{\delta\bar{C}_M(x,t)}\lambda t+\frac{\delta\Gamma}{\delta h_M(x,t)}x_\mu\epsilon_\mu\right\}=0. \quad (65)$$

Let us now examine the consequences of these relations on the renormalized quantities. In a perturbative expansion in the nonlinear coupling constant and in the noise strength, when this exists, the primitive divergences of the theory, i.e., the divergence in the one particle irreducible diagrams obtained as derivatives of the thermodynamic potential Γ, can be easily computed: $\Gamma_{\hat{h}h}$, $\Gamma_{\hat{h}\hat{h}}$, and $\Gamma_{\hat{h}hh}$ have a logarithmic divergence at large impulses. The divergences can be handled by introducing three renormalization functions, to be determined at each order in the perturbative expansion. In particular, we will define a new viscosity ν as a function of the bare one, the corresponding rescaling for the bare frequency and two renormalization factors for the nonlinear coupling constant and for the noise strength:

$$\nu_0=Z\nu, \qquad \omega_0=Z\omega,$$
$$\lambda_0=Z_\lambda\lambda, \qquad D_0=Z_D D. \quad (66)$$

The irreducible correlation functions at fixed ν, ω, λ, and D are finite when $\Lambda\to\infty$ at each order in perturbation theory. Let us now consider the relation (65) for the thermodynamic potential, where we will set $h_m=h$ for simplicity. Taking two derivatives with respect to the fields h and $\hat{h}$ and using the extremal condition

$$\frac{\delta\Gamma(h)}{\delta h(x)}=J_h(x)=0 \quad (67)$$

we obtain the fundamental relation between $\mathcal{G}$ and the three point vertex function $\Gamma^3_{\hat{h}hh}$ [17]:

$$\int dx\delta(x-z)\frac{\partial}{\partial x_\mu}\epsilon_\mu\frac{\delta^2\Gamma}{\delta h(x)\hat{h}(y)}+\frac{\delta^3\Gamma}{\delta h(x)\hat{h}(y)h(z)}x_\mu\epsilon_\mu$$
$$=0 \quad (68)$$

or, in Fourier space:

$$\lambda p_\mu\frac{\partial}{\partial\omega}\mathcal{G}^{-1}(p,-p)+i\frac{\partial}{\partial p_\mu}\Gamma^3_{\hat{h}hh}(p,-p,0)=0. \quad (69)$$

Substituting in (69) the expressions (66) for the renormalized functions we easily obtain

$$Z_\lambda=1. \quad (70)$$

Relation (70) states the invariance of the nonlinear coupling constant under the renormalization procedure. The invariance has been deduced from physical principles in the Burgers equation, using the conservation of the parallel transport term in the hydrodynamic derivative. Here it explicitly follows from Galileian Ward identities. Note that (70) follows from (69) in different renormalization schemes.

Using identity (64) it is also possible to obtain the fundamental relation between the two critical exponents of the model describing the scaling properties of the two point correlation function at large space and time scales:

$$\langle h(\mathbf{x},t)h(\mathbf{x}',t')\rangle\sim\|\mathbf{x}-\mathbf{x}'\|^{2\chi}f\left[\frac{t-t'}{\|\mathbf{x}-\mathbf{x}'\|^z}\right]. \quad (71)$$

The rescaling of the space variable of a factor b, by setting $x\to bx$, implies a corresponding scaling for the time variable, $t\to b^z t$ and for the fields, $h\to b^\chi h$. Substituting these scale relations in (64), we easily obtain

$$\chi+z=2. \quad (72)$$

Relation (72) has been explicitly derived by the Galileian invariance of the KPZ equation.

VI. GALILEIAN WARD IDENTITIES AND CORRELATION AND RESPONSE FUNCTIONS

Some examples of consequences on the physical quantities of the model will now be derived by examining the Ward identities for the free energy W. In order to derive physically meaningful relations, we have to consider explicitly the *boundary conditions* of the dynamical problem and the mean value of the field h. Again, the formalism of the dynamical action allows one to treat this problem easily. As an example let us introduce in the action a quadratic term in the field h, which explicitly breaks the translational symmetry and selects the solution $h_M=0$:

$$S(h,\hat{h},C,\bar{C})=\int d^dxdt\left\{-\frac{D}{2}\,\hat{h}(x,t)^2+\hat{h}(x,t)\left[\frac{\partial}{\partial t}h(x,t)-\nu\Delta h(x,t)-\frac{\lambda}{2}[\boldsymbol{\nabla}h(x,t)]^2\right]\right.$$
$$\left.+\frac{m}{2}\,h(x,t)^2-\bar{C}(x,t)\left[\frac{\partial}{\partial t}C(x,t)-\nu\Delta C(x,t)-\lambda\boldsymbol{\nabla}h(x,t)\boldsymbol{\nabla}C(x,t)\right]\right\}, \quad (73)$$

where m is an arbitrary parameter. It is now easy to derive the Ward identities for the free energy W corresponding to the invariance under Galileian boosts of the new dynamical action. For simplicity, we will skip the fermion fields because they do not contribute when the corresponding sources are taken to be zero:

$$\int d^dxdt\left\{J_h(x,t)\frac{\partial}{\partial x_\mu}\epsilon_\mu\frac{\delta W(J)}{\delta J_h(x,t)}\lambda t+J_{\hat{h}}(x,t)\frac{\partial}{\partial x_\mu}\epsilon_\mu\frac{\delta W(J)}{\delta J_{\hat{h}}(x,t)}\lambda t+J_h(x,t)x_\mu\epsilon_\mu-m\frac{\delta W(J)}{\delta J_h(x,t)}x_\mu\epsilon_\mu\right\}=0. \tag{74}$$

With one derivative with respect to the source $J_h(x,t)$, we obtain an identity for the two point correlation function:

$$y_\mu\epsilon_\mu=m\int d^dxdtx_\mu\epsilon_\mu\frac{\delta^2W(J)}{\delta J_h(x,t)\delta J_h(y,\tau)}, \tag{75}$$

that can be rewritten as

$$y_\mu\epsilon_\mu=m\int d^dxdt[x_\mu-y_\mu+y_\mu]\epsilon_\mu\frac{\delta^2W(J)}{\delta J_h(x,t)\delta J_h(y,\tau)}. \tag{76}$$

Using the symmetry properties of the correlation function, (76) becomes

$$1=m\int d^dxdt\mathcal{C}(x-y,t-\tau). \tag{77}$$

The interesting relation (77) links the value of the parameter m to the value of the two point correlation function $\mathcal{C}$.

Let us now turn to the problem of the effect of a generic boundary condition for the dynamical problem. As an example, we can choose a simple shape of the surface at $t=0$:

$$h(\mathbf{x},0)=0. \tag{78}$$

Under a Galileian boost, condition (78) is changed into

$$h(\mathbf{x}+\lambda\epsilon t,0)_{t=0}=\epsilon\cdot\mathbf{x} \tag{79}$$

and (79) implies an additional term in the variation of the dynamical action. We have to take into account the variation of the field in (x,t) as an effect of the change in the boundary condition in $(y,0)$:

$$h\to h+\epsilon\cdot\mathbf{x}+\int d^dy\frac{\delta h(x,t)}{\delta h(y,0)}\epsilon\cdot\mathbf{y}\quad. \tag{80}$$

Substituting (80) in the functional integral we finally obtain a new set of Ward identities that contains explicitly the boundary condition:

$$\int d^dxdt\left\{J_h(x,t)\frac{\partial}{\partial x_\mu}\epsilon_\mu\frac{\delta W(J)}{\delta J_h(x,t)}\lambda t+J_{\hat{h}}(x,t)\frac{\partial}{\partial x_\mu}\epsilon_\mu\frac{\delta W(J)}{\delta J_{\hat{h}}(x,t)}\lambda t+J_h(x,t)x_\mu\epsilon_\mu-\int d^dyJ_h(x,t)\frac{\delta}{\delta h(y,0)}\frac{\delta W(J)}{\delta J_h(x,t)}y_\mu\epsilon_\mu\right\}=0. \tag{81}$$

One derivative with respect to $J_h(z,\sigma)$ at zero value of the sources gives

$$1=\int d^dy\frac{\delta}{\delta h(y,0)}\frac{\delta W(J)}{\delta J_h(x,t)}, \tag{82}$$

which reads, in Fourier transform,

$$1=\mathcal{R}(k=0,t), \tag{83}$$

where the functional $\mathcal{R}$ is the response function of the system to a change in the boundary conditions:

$$R(x-y,t-\tau)=\frac{\delta}{\delta h(y,\tau)}\frac{\delta W(J)}{\delta J_h(x,t)}. \tag{84}$$

To give a more general interpretation of this response function, one can imagine letting the system evolve for a given time and then take the configuration reached by the field as a new starting condition. A related quantity has been introduced in a different framework [19] in order to study the effects of random initial conditions in domain growth processes and their effects on universality classes. Here relation (83) implies a sort of global conservation law for the bulk quantity corresponding to the response function and represents an interesting link between local and global fluctuations in the mean value of the field. In particular, the effects of a local change in the boundary conditions are damped at long time scales.

In conclusion, the relations obtained here provide a set of interesting constraints for response and correlation functions which can be very useful in any nonperturbative approach. Moreover, this formalism allows one to take into account the effect of the variation of an arbitrary boundary condition.

VII. CONCLUSIONS

In this paper we reviewed the formalism of the dynamical action for a stochastic differential equation for surface growth, the KPZ equation, and, in this framework, we analyzed in detail the consequences of the symmetry properties of the model. We studied in detail the problem of the renormalizability of the theory in a given spatial dimension; in d=2 we discuss the effects of symmetry properties and we give a heuristic demonstration of the renormalizability of the theory. From the Ward identities corresponding to the Galileian invariance of the system we derive the invariance of the nonlinear coupling constant under the renormalization procedure and the fundamental relation between dynamical critical exponents. We obtain a set of Ward identities taking into account the effects of boundary conditions in order to obtain some interesting relationships between physical quantities. The method turns out to be very useful in the study of the consequences of symmetry in the growth model and we plan to apply it in the future to the study of the effects of the particular "local" symmetry on the KPZ equation in $d = 2$ with white noise [20].

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